Differential Geometry

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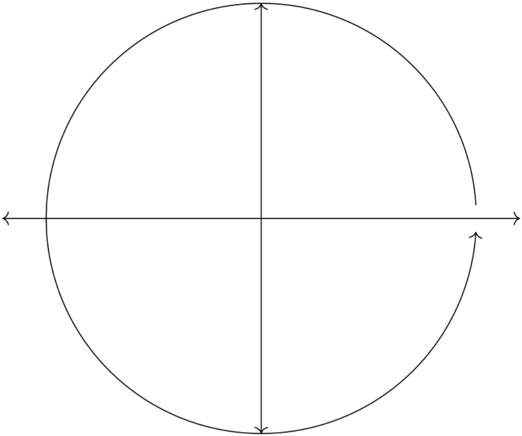
Curves

Definition (Parameterized Differentiable Curve):

We say a curve $\alpha : I \to \mathbb{R}^3$, where I is some open interval $(a, b) \subset \mathbb{R}$, is a differentiable curve if $t \in I$, we can write $\alpha(t) = (x(t), y(t), z(t))$ such that $x(t), y(t), z(t)$ are infinitely differentiable.

Example



Regularity

Definition (Regular Curve):

Given a differentiable curve with a parameterization $\alpha : I \to \mathbb{R}^3$, we say that α is *regular* if $\alpha'(t) \neq 0$.

Example

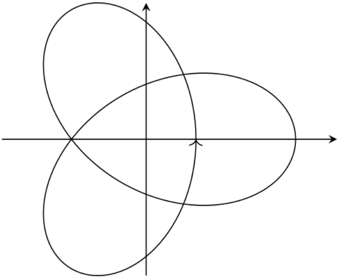


Figure: A trefoil is a regular curve

Arc Length

Definition (Arc Length):

Given a differentiable curve $\alpha : [a, b] \to \mathbb{R}$, we say the arc length L is

$$L = \int_a^b |\alpha'(t)| \mathrm{d}t.$$

Tangent, Normal, Binormal

Let $\alpha : I \to \mathbb{R}^3$ be a differentiable curve parameterized by arc length.

Definition (Curvature):

We define the *curvature*, $\mathbf{k}(s)$ as $|\alpha''(s)|$.

Definition (Tangent, Normal, Binormal):

We define the *tangent* $\mathbf{t}(s)$ as $\alpha'(s)$, the *normal* $\mathbf{n}(s) = \frac{\alpha''(s)}{k(s)}$ and the *binormal* vector $\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s)$.

Frenet frame

Together the tangent, normal and binormal vector create a Frenet frame.
These are related through the following differential equations,

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \mathbf{kn}$$
$$\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s} = -\mathbf{kt} - \tau\mathbf{b}$$
$$\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = \tau\mathbf{n}.$$

Isoperimetric Inequality

Theorem (Isoperimetric Inequality):

Let C, parameterized by $\alpha : I \to \mathbb{R}^2$ be a simple closed plane curve of length ℓ that bounds an area of A, then,

$$\ell^2 - 4\pi A \geq 0$$

where equality holds if and only if C is a circle.

Four Vertex Theorem

Definition (Convex):

We say that a regular plane curve is convex if for every $t \in I = [a, b]$, the trace of $\alpha([a, b])$ lies on one side of the half plane of the tangent line through $\alpha(t)$.

Definition (Vertex):

A vertex of a regular plane curve is a point $t \in I = [a, b]$ such that $\mathbf{k}'(t) = 0$.

Four Vertex Theorem cont.

Theorem (Four Vertex Theorem):

All simple, closed, convex curves have at least 4 vertices.

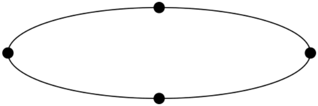


Figure: Ellipse with 4 Vertices

Regular Surfaces

Definition (Regular Surface):

We say that a subset $S \subset \mathbb{R}^3$ is a regular surface if the following conditions are met:

- For each $p \in S$, there exists a neighborhood $V \subset \mathbb{R}^3$ and map $\mathbf{x} : U \to V \cap S$ where $U \subset \mathbb{R}^2$ and open.
- $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)), (u, v) \in U$ is infinitely differentiable.
- $\mathbf{x}$ is a homemorphism, meaning that $\mathbf{x}$ has a well-defined inverse $\mathbf{x}^{-1}$ that is continuous.
- For every $q \in U$, $\mathrm{d}\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.

Examples

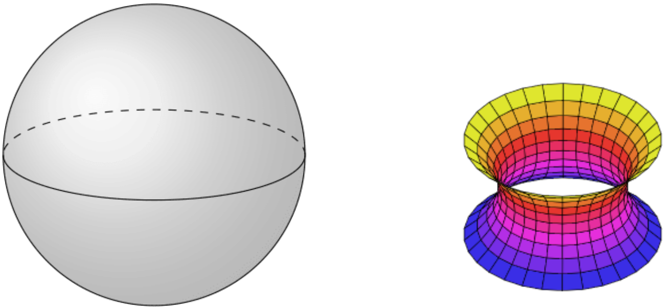


Figure: Sphere and Catenoid(Wikipedia Commons)

Tangent Plane

Definition (Tangent Plane):

We define the tangent plane to be plane spanned by all the tangent vectors at some point p on a regular surface S. We denote this plane as $T_p(S)$.

Notably, $\mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u}, \mathbf{x}_v = \frac{\partial \mathbf{x}}{\partial v}$ creates a basis for the tangent plane.

First Fundamental Form

Definition (First Fundamental Form):

For a regular surface S, the first fundamental form is $\mathrm{I}_p(w) = \langle w, w \rangle_p$ where $\langle , \rangle$ is the standard dot product in $\mathbb{R}^3$, $p \in S$, and $w \in T_p(s)$.

Let $\alpha(t) = \mathbf{x}(u(t), v(t)), t \in V$ be a parameterized curve defined in a neighborhood around p such that $\alpha(0) = \mathbf{x}(u_0, v_0) = p$, then

$$\mathrm{I}_p(\alpha'(0)) = \langle \alpha'(0), \alpha'(0) \rangle$$

First Fundamental Form cont.

$$\begin{aligned} \mathrm{I}_p(\alpha'(0)) &= \langle \mathbf{x}_u u' + \mathbf{x}_v v', \mathbf{x}_u u' + \mathbf{x}_v v' \rangle_p \\ &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle_p (u')^2 + 2\langle \mathbf{x}_u, \mathbf{x}_v \rangle_p u'v' + \langle \mathbf{x}_v, \mathbf{x}_v \rangle_p (v')^2 \\ &= E(u')^2 + 2Fu'v' + G(v')^2. \end{aligned}$$

First Fundamental Form cont.

Definition (First Fundamental Form Coefficients):

This gives rise to the coefficients,

$$E(u,v) = \langle \mathbf{x}_u, \mathbf{x}_u \rangle_p$$
$$F(u,v) = \langle \mathbf{x}_u, \mathbf{x}_v \rangle_p$$
$$G(u,v) = \langle \mathbf{x}_v, \mathbf{x}_v \rangle_p$$

Applications

Definition (Area):

Let $R \subset S$ be a bounded region of a regular surface. Then the area of R is

$$\iint_Q |\mathbf{x}_u \times \mathbf{x}_v| \mathrm{d}u\mathrm{d}v \qquad Q = \mathbf{x}^{-1}(R).$$

Since $|\mathbf{x}_u \times \mathbf{x}_v|^2 + \langle \mathbf{x}_u, \mathbf{x}_v \rangle^2 = |\mathbf{x}_u|^2 |\mathbf{x}_v|^2$, we have

$$A = \iint_Q \sqrt{EG - F^2} \mathrm{d}u\mathrm{d}v.$$

Gauss Map

Definition (Gauss Map):

We can find the normal vector to the tangent plane of a regular surface using,

$$N(q) = \frac{\mathbf{x}_u \times \mathbf{x}_\mathbf{v}}{|\mathbf{x}_u \times \mathbf{x}_\mathbf{v}|}(q) \text{ where } q \in \mathbf{x}(U).$$

Using this, we can impose a map $N : S \to S^2$ where S^2 is the unit sphere. This is called the Gauss Map.

Second Fundamental Form

Definition (Second Fundamental Form):

Using the differential $\mathrm{d}N_p$ (the measure of how much N pulls away from N(p) in some neighborhood around p), we can define the second fundamental form,

$$\mathrm{II}_p(w) = -\langle \mathrm{d}N_p(w), w \rangle_p.$$

Normal Curvature

Definition (Normal Curvature):

If C is a regular curve passing through $p \in S$, k the curvature of C at p, and $\cos\theta = \langle n, N \rangle$ where n is the normal with respect to C, and N is the normal with respect to S. Then the *normal curvature* of $C \in S$ is $k_n = k\cos\theta$.

Normal Curvature cont.

Definition (Principal Curvatures):

We say that the maximum normal curvature is k_1 and minimum as k_2 when considering all directions through p. The corresponding eigenvectors e_1 and e_2 are considered the principal directions.

Applications

Theorem (Euler Formula):

As $k_n = \mathrm{II}_p(w)$, by going along the principal directions, we can show that

$$k_n = k_1 \cos^2(\theta) + k_2 \sin^2(\theta).$$

Applications cont.

Definition:

We say that the Gaussian curvature of S at p is

$$K = k_1 k_2$$

and the mean curvature as

$$H = \frac{k_1 + k_2}{2}.$$

Examples

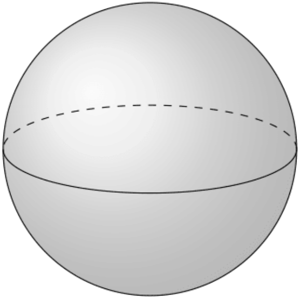


Figure: Unit Sphere: $K = H = 1$

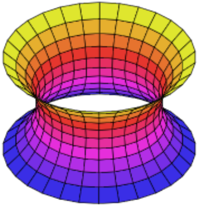


Figure: Catenoid: $H = 0$

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